Higher-Order Traps in Quantum Control Landscapes for Transmon Systems

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Content

In the creation of quantum computers, various physical platforms are used. In this work, a platform based on superconductors is discussed. In Russia, in addition to research in the field of superconductors, areas related to ion traps[1] and photonic technologies[2,3] are actively being developed.

[1] I. V. Zalivako, N. V. Semenin, N. O. Zhadnov, K. P. Galstyan, P. A. Kamenskikh, V. N. Smirnov , A. E. Korolkov, P. L. Sidorov, A. S. Borisenko, Y. P. Anosov, I. A. Semerikov, K. Y. Khabarova, N. N. Kolachevsky, "Quantum computing with trapped ions: principles, achievements, and prospects", Phys. Usp. 68, 552–583 (2025).

[2] N. N. Skryabin, I. V. Kondratyev, I. V. Dyakonov, O. V. Borzenkova, S. P. Kulik, S. S. Straupe, "Two-qubit quantum photonic processor manufactured by femtosecond laser writing", Appl. Phys. Lett. 122, 121102 (2023).

[3] E. V. Moreva, G. A. Maslennikov, S. S. Straupe, S. P. Kulik, "Realization of Four-Level Qudits Using Biphotons", Phys. Rev. Lett. 97, 023602 (2006).

In quantum control of an N-level closed quantum system, one considers Schrödinger equation for a unitary evolution operator U_t^f of the system

$$\frac{dU_t^f}{dt} = -i(H_0 + f(t)H_I)U_t^f, \quad U_t^f\Big|_{t=0} = \mathbb{I},$$

where H_0 and H_I are the free and interaction Hamiltonians and $f(t)$ is a control field.

A wide class of quantum control problems can be expressed as maximizing mean value of some observable $O = O^\dagger$ of the system at some final time $T > 0$ provided that initially the system is in some state ρ_0. In particular, such control problem describes optimal state preparation, relative maximization or minimization of populations of different states, energy optimization, and optimization of other characteristics of the quantum system. Such control problem in general is formulated as maximization of the objective functional

$$\mathcal{F}_O[f] = \langle O \rangle_T = \mathrm{Tr}\left[O U_T^f \rho_0 U_T^{f\dagger}\right] \to \max.$$

The landscape of the quantum control problem is the graph of the objective functional.

The transmon circuit instead of an inductor has a Josephson junction. The parameters of the Josephson junction are[4] $I = I_C \sin\phi$, $I_C = \pi\Delta/(2eR_n)$, and $V = \hbar/(2e) \cdot d\phi/dt$, where I is the current in the Josephson junction, I_C is the critical current, ϕ is the phase difference between the wave functions of superconductors, Δ is the superconducting gap, e is the charge of the electron, R_n is the normal barrier resistance, V is the voltage on junction, and $\hbar$ is the reduced Plank constant.

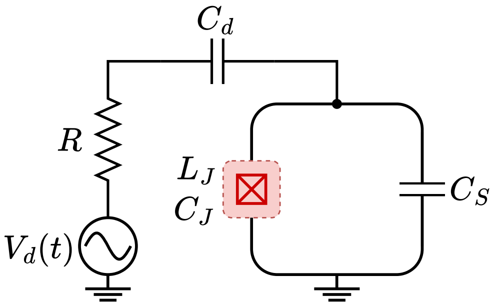

Figure 1: Circuit with capacitive coupling of a microwave drive line.

[4] Josephson, B.: Possible new effects in superconductive tunnelling. Physics Letters 1, 251–253 (1962).

Transmon System

The free Hamiltonian for the transmon circuit[5,6]

$$\hat{H}_{tr} = 4E_C\hat{n}^2 + E_J(1 - \cos\hat{\phi}),$$

where $\hat{n}$ is a Cooper number operator, $\hat{\phi}$ is a phase operator, $E_C = e^2/(2C_{tr}) \neq 0$ is the energy of total capacitance of the transmon $C_{tr} = C_S + C_J$, consisting of the shunt capacitance C_S and self-capacitance of the junction C_J, $E_J = I_C\hbar/2e = I_C\Phi_0/2\pi$ is Josephson energy and Φ_0 is the magnetic flux quantum.

The operators $\hat{n}$ и $\hat{\phi}$ satisfy the canonical commutation relations $[\hat{\phi}, \hat{n}] = i\mathbb{I}$. In terms of the creation and annihilation operators they are written as

$$\hat{n} = i \cdot n_{\text{zpf}}\left(\hat{a}^\dagger - \hat{a}\right), \quad \hat{\phi} = \phi_{\text{zpf}}\left(\hat{a}^\dagger + \hat{a}\right),$$

where $n_{\text{zpf}} = \sqrt[4]{E_J/32E_C}$ и $\phi_{\text{zpf}} = \sqrt[4]{2E_C/E_J}$ are the zero-point fluctuations of the charge and phase variables.

[5] J. Koch et al., "Charge-insensitive qubit design derived from the Cooper pair box", Phys. Rev. A 76, 042319 (2007).

[6] P. Krantz et al., "A quantum engineer's guide to superconducting qubits", Appl. Phys. Rev. 6, 021318 (2019).

Transmon System

The transmon system satisfies the condition $E_J/E_C \gg 1$ (in practice $E_J/E_C \geq 20$)[7], in which the zero-point fluctuation of the phase variable is very small. That allows to obtain an approximate Hamiltonian of the transmon system

$$\hat{H}_{tr} = 4E_C\hat{n}^2 + \frac{1}{2}E_J\hat{\phi}^2 - \frac{1}{24}E_J\hat{\phi}^4.$$

Then the approximate Hamiltonian of the transmon can be written terms of the creation and annihilation operators

$$\hat{H}_{tr} = (\hbar\omega_r - E_C)\,\hat{a}^\dagger\hat{a} - \frac{E_C}{2}\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a},$$

where $\omega_r = \sqrt{8E_JE_C}/\hbar = 1/\sqrt{L_JC_{tr}} \neq 0$ is the resonant frequency. The spectrum of this Hamiltonian is

$$E_n = (\hbar\omega_r - E_C)n - E_C\frac{n^2 - n}{2}.$$

[7] J. Koch et al., "Charge-insensitive qubit design derived from the Cooper pair box", Phys. Rev. A 76, 042319 (2007).

Transmon System

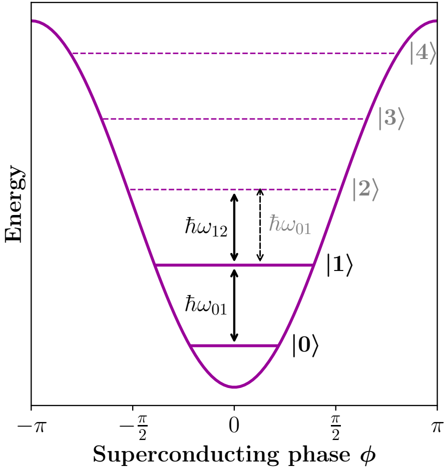

Figure 2: Energy spectrum of the transmon qubit. Due to the anharmonicity, the transition energy $\hbar\omega_{01}$ between states $|0\rangle$ and $|1\rangle$ differs from the transition energy $\hbar\omega_{12}$ between states $|1\rangle$ and $|2\rangle$. In general, the transition energy between adjacent levels decreases as levels increase. This allows to limit the system to only some lowest energy levels.

Transmon System with Control

The complete Hamiltonian for a transmon system with control

$$\hat{H} = \hat{H}_{tr} + f(t)\hat{V}_{tr},$$

where $f(t) = \frac{C_d}{C_\Sigma} Q_{\mathrm{zpf}} V_d(t)$ is the control, $Q_{\mathrm{zpf}} = \sqrt{\hbar/(2L\omega_r)}$, and $\hat{V}_{tr} = -i\left(\hat{a} - \hat{a}^\dagger\right)$ is the interaction Hamiltonian of the transmon with the control. The total capacitance C_Σ for the transmon system has the form $C_\Sigma = C_{tr} + C_d$.

Restriction of the initial full infinite-dimensional Hamiltonian to a subspace of dimension N. Consider the first N lowest energy states of the transmon system with the free and control Hamiltonians

$$\hat{H}_{tr_N} = \sum_{n=0}^{N-1} \Big((\hbar\omega_r - E_C)n - \frac{1}{2} E_C n(n-1) \Big) |n\rangle\langle n|,$$

$$\hat{V}_{tr_N} = -i \sum_{n=0}^{N-2} \sqrt{n+1} \Big(|n\rangle\langle n+1| - |n+1\rangle\langle n| \Big).$$

Strong Regularity for Transmon System

Anharmonicity is important for avoiding symmetries in the energy level structure. For a quantum system anharmonicity follows from the condition of strong regularity.

Definition 1

A *strongly regular quantum system* is a quantum system with different energy levels and different Bohr frequencies of the free Hamiltonian.

Let us assume that some Bohr frequencies coincide, $E_{k\to n} = E_{l\to m}$ for $n + l \neq k + m$. Expressing $E_{k\to n}$ and $E_{l\to m}$ using formula

$$E_{k\to n} = (\hbar\omega_r - E_C)(n-k) - \frac{1}{2}E_C(n^2 - n - k^2 + k),$$

we get

$$\frac{E_J}{E_C} = \frac{1}{32}\left(\frac{n^2 + l^2 - k^2 - m^2}{n + l - k - m} + 1\right)^2.$$

Introduce the set $P_N := \{(k,l,m,n) : k,l,m,n \in \{0,\dots,N-1\}, n+l \neq k+m\}$. Let $G[N] = \max\left\{\frac{1}{32}\left(\frac{n^2+l^2-k^2-m^2}{n+l-k-m}+1\right)^2 : (k,l,m,n) \in P_N\right\}$.

Proposition 1

If $E_J/E_C > G[N]$, then N-level transmon system is strongly regular.

For a transmon system $E_J/E_C \gg 1$, and in practice $E_J/E_C \geq 20$. Therefore, the three and four-level transmon systems are strongly regular by default.

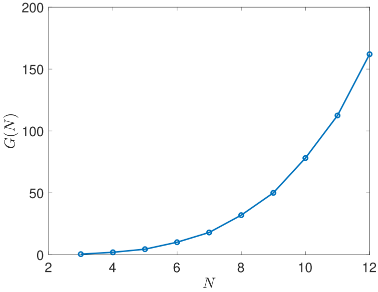

Figure 3: Plot of the function $G(N)$ for $N = 3,\dots,12$. If $E_J/E_C > G(N)$, then the transmon system is strongly regular.

A closed quantum system (H_0, H_I), whose dynamics is described by the Schrödinger equation, is called *completely controllable*[8] if there exists a time $T_{\min}$ such that for any time $T > T_{\min}$ and for any $U \in \mathrm{U(N)}$ there exists a control $f \in L_2([0,T],\mathbb{R})$ such that $U = e^{i\alpha}U_T^f$, where $\alpha \in [0, 2\pi)$.

From the article[9] we can conclude, that strongly regular system with the chained Hamiltonian

$$V = \sum_{n=0}^{N-2} \Big(v_{n,n+1}|n\rangle\langle n+1| + v^*_{n,n+1}|n+1\rangle\langle n| \Big),$$

where all $v_{n,n+1} \neq 0$, is complete controllable. The transmon system $(\hat{H}_{tr_N}, \hat{V}_{tr_N})$ is a special case of a strongly regular system with a chained interaction Hamiltonian, hence is completely controllable. Note that a controllability of anharmonic quantum systems with chained interaction Hamiltonians with real matrix coefficients was studied.[10] The complete controllability of the transmon system can also be verified with a slight modification of the proofs in that article.

[8] P. De Fouquieres and S. G. Schirmer, "A closer look at quantum control landscapes and their implication for control optimization", IDAQP 16, 1350021 (2013).

[9] C. Altafini, "Controllability of quantum mechanical systems by root space decomposition of su(n)", J. Math. Phys. 57, 2051–2062 (2002).

[10] S. Schirmer, H. Fu, A. Solomon, "Complete controllability of quantum systems" Phys. Rev. A. 63, 063410 (2001).

Trap is a local but not global maximum (minimum) of the objective functional $\mathcal{F}$ in the problem of quantum system control.[11] Traps of the 3-rd order were considered for the Λ-atom.[12] The *n-th order trap* is defined by decomposition of the objective functional in a Taylor series up to the n-th order.[13]

A control $f \in \mathfrak{H}^0 := L_2([0,T],\mathbb{R})$ is a trap of n-th order of the objective functional $\mathcal{F}_O$, if $\mathcal{F}_O$ does not reach its global maximum at the control f and the Taylor expansion of the objective functional at f has the form[14]

$$\mathcal{F}_O(f+\delta f) = \mathcal{F}_O(f) + \sum_{j=2}^{n} \frac{1}{j!}\mathcal{F}_O^{(j)}(f)(\delta f,\ldots,\delta f) + o(\|\delta f\|^n), \quad \|\delta f\| \to 0,$$

where the polynomial $R(\delta f) = \sum\limits_{j=2}^{n} \frac{1}{j!}\mathcal{F}_O^{(j)}(f)(\delta f,\ldots,\delta f)$ satisfies the conditions:

1. there exists $\delta f \in \mathfrak{H}^0$ such that $R(\delta f) < 0$;
2. for any $\delta f \in \mathfrak{H}^0$ there exists $\epsilon > 0$ such that $R(t\delta f) \leq 0$ for all $t \in (-\epsilon, \epsilon)$.

[11] H. A. Rabitz, M. M. Hsieh, and C. M. Rosenthal, ”Quantum optimally controlled transition landscapes”, Science 303, 1998–2001 (2004).

[12] A. N. Pechen, D. J. Tannor, ”Are there Traps in Quantum Control Landscapes?”, Phys. Rev. Lett. 106, 120402 (2011).

[13] A. Pechen, D. Tannor, ”Quantum control landscape for a Lambda-atom in the vicinity of second-order traps”, Isr. J. Chem. 52, 467–472 (2012).

[14] B. O. Volkov, A. N. Pechen, ”Higher-order traps for some strongly degenerate quantum control systems”, Russ. Math. Surv. 2, 390–392 (2023).

Third- and Fifth-Order Traps

The three-level approximation of the transmon system has a forbidden transition between one pair of levels and thereby corresponds to the class of three-level Λ-type quantum systems. Using the results of the article[15] shows, that for a three-level transmon system, the null control is a trap of the third order.

Proposition 2

For the three-level transmon system, the null control $f_0 \equiv 0$ is a trap of the third order for the objective functional $\mathcal{F}_O$ with any observable of the form $O = \lambda_0|0\rangle\langle 0| + \lambda_1|1\rangle\langle 1|$, where $\lambda_0 > 0$ and $\lambda_1 < 0$, for the initial state $\rho_0 = |2\rangle\langle 2|$, and for any $T \geq T_{\min}$.

Consider a completely controllable closed quantum system with the free Hamiltonian $H_0 = \sum_{k=0}^{N-1} h_k|k\rangle\langle k|$ and a chained interaction Hamiltonian V.

Proposition 3

For a controllable four-level system (H_0, V) with a chained interaction Hamiltonian, the null control $f_0 \equiv 0$ is a trap of the fifth order for the objective functional $\mathcal{F}_O$ for any $O = \sum_{k=0}^{2} \lambda_k|k\rangle\langle k|$, where $0 < \lambda_0$ and $\lambda_k < 0$ for $k \in \{1, 2\}$, for the initial state $\rho_0 = |3\rangle\langle 3|$, and for any $T \geq T_{\min}$.

[15] B. Volkov, A. Myachkova, A. Pechen, "Phenomenon of a stronger trapping behavior in Λ-type quantum systems with symmetry", Phys. Rev. A 111, 022617 (2025).

To study the landscape of quantum control with objective functional maximizing the average value of the observable the method[16] based on GRAPE algorithm[17] is used.

In this approach, we consider piecewise-constant controls of the form

$$f_C(t) = \sum_{k=1}^{M} c_k \chi_{(t_k, t_{k+1}]}(t),$$

where $C = (c_1, \ldots, c_M) \in \mathbb{R}^M$ is a M-dimensional control vector, $\chi_{(t_k, t_{k+1}]}(t)$ is the characteristic function of the half-interval $(t_k, t_{k+1}]$, $t_k = \Delta t(k-1)$, and $\Delta t = T/M$. The objective functional $\mathcal{F}_O(f)$ is replaced by the objective function $\mathcal{J}_O \colon \mathbb{R}^M \to \mathbb{R}$, where

$$\mathcal{J}_O(C) = \mathcal{F}_O(f_C) = \mathrm{Tr}[U_T^{f_C} \rho_0 (U_T^{f_C})^\dagger O].$$

[16] T. Schulte-Herbrüggen et al., "Optimal control for generating quantum gates in open dissipative systems", J. Phys. B: At. Mol. Opt. Phys. 44, 154013 (2011).

[17] N. Khaneja et al., "Optimal control of coupled spin dynamics: design of NMR pulse sequences by gradient ascent algorithms", J. Magn. Reson. 172, 296–305 (2005).

Gradient of the objective function is determined by the partial derivatives, for which we use the linear in Δt approximation to define the linear in Δt approximate expression $\mathrm{grad}_{\mathrm{lin}}\mathcal{J}_O(C)$ for the gradient

$$\frac{\partial \mathcal{J}_O(C)}{\partial c_k} \approx 2\Delta t \times \mathrm{Im}\Big[\mathrm{Tr}\Big(W_k^\dagger V W_k \rho_0 W_M^\dagger O W_M\Big)\Big] =: (\mathrm{grad}_{\mathrm{lin}}\mathcal{J}_O(C))_k,$$

where $W_k = U_k U_{k-1} \dots U_2 U_1$ and $U_k = e^{-i(H_0 + c_k V)\Delta t}$ for $k = 1, \dots, M$.

We randomly, with a uniform distribution, generate a sufficiently large number L of initial control vectors in a hypercube $[-l, l]^M$. Then starting at each such initial control, we apply GRAPE algorithm with the approximate gradient and a fixed step size ε, so that at i-th iteration the control is updated as

$$C_{i+1} = C_i + \varepsilon \cdot \mathrm{grad}_{\mathrm{lin}}\mathcal{J}_O(C_i).$$

The algorithm stops either if the objective value $J_{\max} = 1 - I_{\mathrm{err}}$ for sufficiently small I_{err} is obtained, or if a maximal number of iterations K_{stop} is reached. If the algorithm stops due to the second criterion (maximal number of iterations K_{stop} is reached), and the objective value $J_{\max}$ is not obtained, we call this run as failed run. Otherwise we call the run as successful.

Numerical Analysis of the Control Landscape in a Vicinity of the Null Control Five-Order Trap

For the problem of maximizing the average of the quantum observable, numerical simulation is performed in the vicinity of the null control for four-level transmon system with parameters $E_J/h = 28.6$ GHz and $E_C/h = 0.292$ GHz.[18] The corresponding four-level approximating Hamiltonians approximately are

$$\hat{H}_{tr_4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 7.88 & 0 & 0 \\ 0 & 0 & 15.47 & 0 \\ 0 & 0 & 0 & 22.77 \end{pmatrix}, \quad \hat{V}_{tr_4} = i \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -\sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}.$$

For simulations we use $O = \mathrm{diag}(1, -5, -5, 0)$, $T = 20$, $M = 100$, $\varepsilon = 0.02$, $I_{\text{err}} = 10^{-3}$, $K_{\text{stop}} = 2000$, and $L = 1000$.

[18] H. Paik et al., "Observation of high coherence in Josephson junction qubits measured in a three-dimensional circuit QED architecture", Phys. Rev. Lett. 107, 240501 (2011).

Numerical Analysis of the Control Landscape in a Vicinity of the Null Control Five-Order Trap

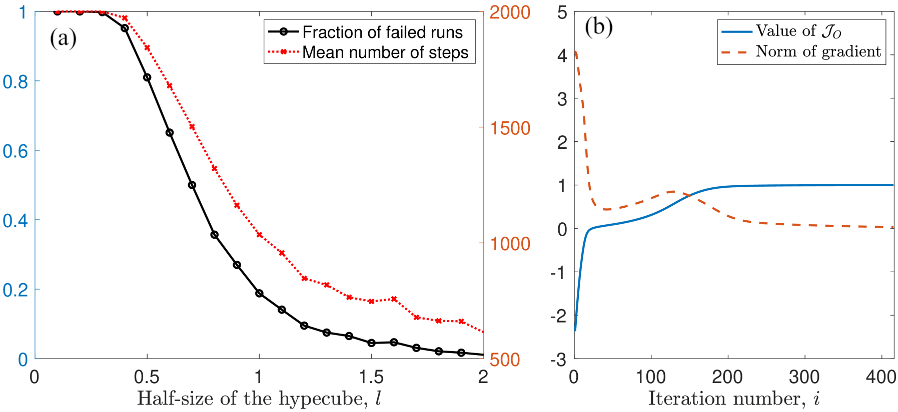

Figure 4: (a) Fraction of failed runs (left vertical scale) and mean number of GRAPE iteration steps (right vertical scale) vs $l = 0.1, 0.2, \ldots, 2$ for the transmon system. For each value of l, $L = 1000$ runs are performed. (b) An example behaviour of the objective value $\mathcal{J}_O(C_i)$ and of the norm of the approximate gradient $\|\mathrm{grad}_{\mathrm{lin}}(C_i)\|$ vs iteration number i (416 iterations were performed) for $l = 2$, when almost all $L = 1000$ runs were successful.

Conclusions

- The strong regularity and complete controllability of the transmon system was studied.
- The quantum control landscape for maximizing the mean value of a quantum observable for three- and four-level approximations of a transmon superconducting system has been studied.
- It was found that the null control is a trap of the third order for the three-level transmon approximation and a trap of the fifth order for the four-level approximation.
- Numerical simulations using GRAPE algorithm show that in a close vicinity of the null control (higher-order trap) optimization is inefficient, whereas with the increase of the average distance between the initial control and the null control optimization becomes more and more efficient.

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Thank You For Attention!